DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

SAVAGE IN THE MARKET

Federico Echenique

Kota Saito

SOCIAL SCIENCE WORKING PAPER 1380

October 2013

Savage in the Market

Federico Echenique Kota Saito

Abstract

We develop a behavioral axiomatic characterization of Subjective Expected Utility (SEU) under risk aversion. Given is an individual agent's behavior in the market: assume a finite collection of asset purchases with corresponding prices. We show that such behavior satisfies a "revealed preference axiom" if and only if there exists a SEU model (a subjective probability over states and a concave utility function over money) that accounts for the given asset purchases.

JEL classification numbers: D81, D11

Key words: Subjective Expected Utility; Risk Aversion; Revealed Preference; Testable Implications

Savage in the Market*

Federico Echenique Kota Saito

1 Introduction

Savage's (1954) characterization of subjective expected utility (SEU) has been called the "crowning achievement of single-person decision theory" (Kreps, 1988). Savage describes the behaviors that are consistent with SEU, where behavior is modeled as a preference relation over acts (i.e state-contingent consequences). The result we present in this paper also describes the behaviors that are consistent with SEU, but *we focus on economic or market behavior.*

Savage (1954) obtains a set of axioms that is necessary and sufficient for SEU. He takes as given a state space and a set of consequences; an analyst observes the decision maker's preference relation. The analyst observes all pairwise comparisons of acts that the decision maker might make. Such comparisons are compatible with SEU if and only if they satisfy Savage's axioms. Another way of explaining Savage's contribution is that his result describes all possible "paradoxes" of rational behavior that one can generate in a certain class of thought experiments. For example, the thought experiments proposed by Ellsberg (1961) can be traced to the violation of one of Savage's axioms (the sure-thing principle).

Our framework differs from Savage. We too take as given a state space, but focus on behavior consisting of purchases of state-contingent monetary assets. An analyst observes a finite collection of purchases with corresponding prices. Such data are standard in the theory of revealed preference.

Our main result is that a certain axiom describes the data that are consistent with *risk averse SEU preferences.* The axiom is a generalization of the simplest implication of risk aversion on the relation between prices and quantities: that demand slopes down. Our axiom constraints quantities and prices in a way that generalizes downward-sloping demand, but accounts for the different unobservable components in SEU. Section 3 has an informal derivation of the axiom, together with a formal statement of our main result.

*We thank Kim Border and Chris Chambers for inspiration, comments, and advice.

Similarly to Savage's, our result can be said to describe all paradoxes of SEU. The paradoxes would be observable from market experiments, or market thought experiments, in which subjects are presented with financial decisions. Any collection of financial decisions that violate our axiom cannot be compatible with SEU, and all incompatible decisions are generated by negating our axiom.

Our paper also discusses quasilinear SEU. We include this model for two reasons. One is that quasilinear SEU is heavily used in economics, and therefore of independent interest. The other is pedagogical. The axiom for quasilinear SEU turns out be an obvious special case of the axiom for SEU, and the abalysis is much simpler.

As far as we know, the only precedent to our paper that is related to SEU is Epstein (2000). Epstein's setup is the same as ours: in particular, he assumes data on state-contingent asset purchases, and that probabilities are subjective and unobserved. He presents a necessary condition for market behavior to be consistent with probabilistic sophistication. The model of probabilistic sophistication is more general than SEU, and Epstein's necessary condition is easily seen to be a special case of our axiom.

A few papers study von-Neumann Morgenstern objective expected utility in a similar revealed preference framework to ours. Green and Srivastava (1986), Varian (1983) and Varian (1988) carry out the same exercise as we do, but they take probabilities (priors) as objective and as part of the observed data on the agent's behavior. In contrast, we assume that probabilities are subjective and unobserved. Varian (1988) focuses on empirically recovering an agent's attitude towards risk. Like Green and Srivastava, he assumes that data on objective probabilities is available, along with prices and asset purchases. Varian (1983) raises the possibility of using his results when probabilities are unobserved, but doing so would require establishing the existence of values of the unobservables for which the data satisfy Varian's test. He does not provide a combinatorial axiom describing the data that are SEU rational.

2 Setup

We use the following notational conventions: For vectors $x, y \in \mathbf{R}^n$, $x \leq y$ means that $x_i \leq y_i$ for all $i = 1, \ldots, n$; $x < y$ means that $x \leq y$ and $x \neq y$; and $x \ll y$ means that $x_i < y_i$ for all $i = 1, \ldots, n$.

Definition 1. *A* dataset *is a collection* $(x^k, p^k)_{k=1}^K$ *where for all* k $x^k, p^k \in \mathbf{R}^S_{++}$, *and* $x^k_s \neq x^{k'}_{s'}$ *if* $(k, s) \neq (k', s')$.

We assume a finite number of states S. A state-contingent payoff, or *monetary act* , is a vector $x \in \mathbf{R}^S_+$. The interpretation of a dataset is as follows. There are K observations, indexed by $k = 1, \ldots, K$. Each observation consists of a monetary act x^k purchased at some vector of strictly positive state prices p^k.

The assumption that $x^k_s \neq x^{k'}_{s'}$ if $(k, s) \neq (k', s')$ simplifies the analysis. The essence of our results is true without it: see Section 4.1.

Definition 2. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* subjective expected utility rational *(SEU rational) if there is a vector* $\mu \in \mathbf{R}^S_{++}$ *with* $\sum_{s=1}^S \mu_s = 1$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k,

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x^k_s).$$

Definition 3. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* quasilinear subjective expected utility rational *(QL-SEU rational) if there is a vector* $\mu \in \mathbf{R}^S_{++}$ *with* $\sum_{s=1}^S \mu_s = 1$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k,

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{s \in S} \mu_s u(y_s) - p^k \cdot y \leq \sum_{s \in S} \mu_s u(x^k_s) - p^k \cdot x^k.$$

3 Main Results

We start by discussing QL-SEU rationality because the analysis is similar, but simpler, to the analysis of SEU rationality.

3.1 QL-SEU rationality

A QL-SEU maximizing agent solves the problem

$$\max_{x \in \mathbf{R}^S_+} \sum_{s \in S} \mu_s u(x_s) - p \cdot x$$

when faced with prices p. Suppose that the function u is continuously differentiable (an assumption that turns out to be without loss of generality). The first-order condition for the agent's maximization problem is

$$\mu_s u'(x_s) = p_s.$$

So if a dataset $(x^k, p^k)_{k=1}^K$ is QL-SEU rational, the prior μ and utility u must satisfy the above first order condition for each x^k_s and p^k_s; that is: $\mu_s u'(x^k_s) = p^k_s$.

The first-order condition has an immediate implication for consumption at a given state. If $x^k_s > x^{k'}_s$ then the concavity of u implies that we must have $p^k_s \geq p^{k'}_s$. This implication amounts to saying that "state s demand must slope down."

We cannot draw a similar conclusion when comparing $x^k_s > x^{k'}_{s'}$ with $s \neq s'$ because the effect of the different priors μ_s and $\mu_{s'}$ may interfere with the effect of prices on demand. From the first-order conditions:

$$\frac{u'(x^{k'}_{s'})}{u'(x^k_s)} = \frac{\mu_s}{\mu_{s'}} \frac{p^{k'}_{s'}}{p^k_s}.$$

Now, the concavity of u and $x_s^k > x_{s'}^{k'}$ implies that

$$\frac{\mu_s}{\mu_{s'}} \frac{p_{s'}^{k'}}{p_s^k} \leq 1,$$

but the priors μ are unobservable so we cannot conclude anything about the observable $\frac{p_{s'}^{k'}}{p_s^k}$.

There is, however, one further implication of QL-SEU and the concavity of u. We can consider a sequence of pairs $(x_s^k, x_{s'}^{k'})$ chosen such that when we divide first-order conditions all the priors cancel out. For example, consider

$$x_{s_1}^{k_1} > x_{s_2}^{k_2} \text{ and } x_{s_2}^{k_3} > x_{s_1}^{k_4}.$$

By manipulating the first-order conditions we obtain that:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_2})} \cdot \frac{u'(x_{s_2}^{k_3})}{u'(x_{s_1}^{k_4})} = \left(\frac{\mu_{s_2}}{\mu_{s_1}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \right) \cdot \left(\frac{\mu_{s_1}}{\mu_{s_2}} \frac{p_{s_2}^{k_3}}{p_{s_1}^{k_4}} \right) = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_1}^{k_4}}$$

Notice that the pairs $(x_{s_1}^{k_1}, x_{s_2}^{k_2})$ and $(x_{s_2}^{k_3}, x_{s_1}^{k_4})$ have been chosen so that the priors μ_{s_1} and μ_{s_2} would cancel out. Now the concavity of u and the assumption that $x_{s_1}^{k_1} > x_{s_2}^{k_2}$ and $x_{s_2}^{k_3} > x_{s_3}^{k_4}$ imply that the product of the prices $\frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_1}^{k_4}}$ cannot exceed 1. Thus, we obtain an implication of QL-SEU for prices, an observable entity.

In general, the assumption of QL-SEU rationality will require that, for any collection of sequences as above (appropriately chosen so that priors will cancel out) the product of the ratio of prices cannot exceed 1. Formally,

Axiom 1. *For any sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

1. $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ *for all i;*
2. *each s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right):*

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

Our first result is that this necessary condition turns out be sufficient as well. The axiom seems weak in the following sense. *The axiom imposes a particular behavior under circumstances in which priors do not matter–when the priors cancel out as above.* It tell us to focus on circumstances in which the priors can be ignored, and only constrains behavior in such circumstances.

Theorem 1. $(x^k, p^k)_{k=1}^K$ *is QL-SEU rational if and only if it satisfies Axiom 1.*

3.2 SEU rationality

We now discuss the axiom for SEU rationality. In this case, the first order conditions contain three unobservables. The conditions involve not only priors and marginal utilities but also Lagrange multipliers:

$$\mu_s u'(x_s) = \lambda p_s.$$

Hence, from the first-order conditions:

$$\frac{u'(x_{s'}^{k'})}{u'(x_s^k)} = \frac{\mu_s}{\mu_{s'}} \frac{\lambda^{k'}}{\lambda^k} \frac{p_{s'}^{k'}}{p_s^k}.$$

We can repeat the idea used for QL-SEU rationality, but the sequences must be chosen so that not only priors but also Lagrange multipliers cancel out. For example, consider

$$x_{s_1}^{k_1} > x_{s_2}^{k_2}, x_{s_2}^{k_3} > x_{s_3}^{k_1}, \text{ and } x_{s_3}^{k_2} > x_{s_1}^{k_3}.$$

By manipulating the first-order conditions we obtain that:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_2})} \cdot \frac{u'(x_{s_2}^{k_3})}{u'(x_{s_3}^{k_1})} \cdot \frac{u'(x_{s_3}^{k_2})}{u'(x_{s_1}^{k_3})} = \left(\frac{\mu_{s_2}}{\mu_{s_1}} \frac{\lambda^{k_1}}{\lambda^{k_2}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}}\right) \cdot \left(\frac{\mu_{s_3}}{\mu_{s_2}} \frac{\lambda^{k_3}}{\lambda^{k_1}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}}\right) \cdot \left(\frac{\mu_{s_1}}{\mu_{s_3}} \frac{\lambda^{k_2}}{\lambda^{k_3}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}\right) = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}$$

Notice that the pairs $(x_{s_1}^{k_1}, x_{s_2}^{k_2})$, $(x_{s_2}^{k_3}, x_{s_3}^{k_1})$, and $(x_{s_3}^{k_2}, x_{s_1}^{k_3})$ have been chosen so that the priors μ_{s_1}, μ_2, and μ_{s_3} and the Lagrange multipliers $\lambda^{k_1}, \lambda^{k_2}$, and λ^{k_3} would cancel out. Now the concavity of u and the assumption that $x_{s_1}^{k_1} > x_{s_2}^{k_2}$, $x_{s_2}^{k_3} > x_{s_3}^{k_1}$, and $x_{s_3}^{k_2} > x_{s_1}^{k_3}$ imply that the product of the prices $\frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}$ cannot exceed 1. Thus, we obtain an implication of SEU for prices again, an observable entity.

In general, the assumption of SEU rationality will require that, for any collection of sequences as above (appropriately chosen so that priors and Lagrange multipliers will cancel out) the product of the ratio of prices cannot exceed 1. Formally,

Axiom 2. *For any sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

1. $x_{s_i}^{k_i} \geq x_{s_i'}^{k_i'}$ *and* $(k_i, s_i) \neq (k_i', s_i')$ *for all* i*;*
2. *each* s *appears as* s_i *(on the left of the pair) the same number of times it appears as* s_i' *(on the right);*
3. *each* k *appears as* k_i *(on the left of the pair) the same number of times it appears as* k_i' *(on the right):*

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

Note that Axiom 2 is different from Axiom 1 only in the third requirement for the sequence. The main finding of our paper is that this necessary condition is sufficient as well.[1]

Theorem 2. *$(x^k, p^k)_{k=1}^K$ is SEU rational if and only if it satisfies Axiom 2.*

3.2.1 Relationship with Epstein (2000)

As mentioned, Epstein (2000) obtains a necessary condition for probability sophistication. We can phrase the condition as follows. For any sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^2$ satisfying (1), (2) and (3) in Axiom 2, with $k_1 = k'_1 \neq k_2 = k'_2$, it must hold that $p^{k'_1}_{s'_1} \leq p^{k_1}_{s_1}$ or $p^{k'_2}_{s'_2} \leq p^{k_2}_{s_2}$.[2]

For such a sequence, Axiom 2 requires that

$$\frac{p^{k_1}_{s_1}}{p^{k'_1}_{s_2}} \frac{p^{k_2}_{s_1}}{p^{k'_2}_{s_2}} \leq 1,$$

so Axiom 2 clearly implies the necessary condition for probability sophistication.

3.2.2 Axiom 2 implies WARP

The classical necessary condition for rational choice is the weak axiom of revealed preference (WARP). By Theorem 2, any dataset that satisfies Axiom 2 must also satisfy WARP. It is instructive to present a direct proof of this fact.

Definition 4. *A dataset $(x^k, p^k)_{k=1}^K$ satisfies WARP if there is no k and k' such that $p^k \cdot x^k \geq p^k \cdot x^{k'}$ and $p^{k'} \cdot x^{k'} > p^{k'} \cdot x^k$.*

Proposition 1. *If $(x^k, p^k)_{k=1}^K$ satisfies Axiom 2, then $(x^k, p^k)_{k=1}^K$ satisfies WARP.*

Proof. Suppose, towards a contradiction, that D satisfies Axiom 2 but that it violates WARP. Then there is k and k' such that $p^k \cdot x^k \geq p^k \cdot x^{k'}$ and $p^{k'} \cdot x^{k'} > p^{k'} \cdot x^k$. It cannot be the case that $x^k_s \geq x^{k'}_s$ for all s, so the set $S_1 = \{s : x^k_s < x^{k'}_s\}$ is nonempty. Choose $s^* \in S_1$ such that

$$\frac{p^{k'}_{s^*}}{p^k_{s^*}} \geq \frac{p^{k'}_s}{p^k_s} \text{ for all } s \in S_1.$$

Now, $p^k \cdot x^k \geq p^k \cdot x^{k'}$ implies that

$$(x^k_{s^*} - x^{k'}_{s^*}) \geq \frac{-1}{p^k_{s^*}} \sum_{s \neq s^*} p^k_s (x^k_s - x^{k'}_s).$$

[1] In Echenique and Saito (2013), we study intertemporal decision making problem by using the same setup with the interpretation of S as the set of periods. In the paper, we obtain one axiom that is necessary and sufficient for exponential-discounting utility model. The axiom is the same as Axiom 2 except in condition (2). In the axiom, we require that $\sum s_i = \sum s'_i$, instead of condition (2).

[2] The condition (3) requires that $s_1 = s'_2 \neq s_2 = s'_1$.

We also have that $p^{k'} \cdot x^{k'} > p^{k'} \cdot x^k$, so

$$\begin{aligned} 0 &> \sum_{s \neq s^*} p_s^{k'} (x_s^k - x_s^{k'}) + p_{s^*}^{k'} (x_{s^*}^k - x_{s^*}^{k'}) \\ &\geq \sum_{s \neq s^*} p_s^{k'} (x_s^k - x_s^{k'}) + \frac{-p_{s^*}^{k'}}{p_{s^*}^k} \sum_{s \neq s^*} p_s^k (x_s^k - x_s^{k'}) \\ &= \underbrace{\sum_{s \notin S_1} p_s^{k'} (1 - \frac{p_{s^*}^{k'} p_s^k}{p_{s^*}^k p_s^{k'}})(x_s^k - x_s^{k'})}_{A} + \underbrace{\sum_{s \in S_1 \setminus \{s^*\}} p_s^{k'} (1 - \frac{p_{s^*}^{k'} p_s^k}{p_{s^*}^k p_s^{k'}})(x_s^k - x_s^{k'})}_{B} . \end{aligned}$$

We shall prove that $A \geq 0$ and that $B \geq 0$, which will yield the desired contradiction.

For all $s \notin S_1$ we have that $(x_s^k - x_s^{k'}) \geq 0$. Then Axiom 2 implies that

$$\frac{p_{s^*}^{k'} p_s^k}{p_{s^*}^k p_s^{k'}} \leq 1,$$

as $x_{s^*}^k < x_{s^*}^{k'}$ so that the sequence $\{(x_{s^*}^{k'}, x_{s^*}^k),\ (x_s^k, x_s^{k'})\}$ satisfies (1), (2), and (3) in the axiom. Hence $A \geq 0$.

Now consider B. By definition of s^*, we have that $\frac{p_{s^*}^{k'} p_s^k}{p_{s^*}^k p_s^{k'}} \geq 1$ for all $s \in S_1$. Then, $(x_s^k - x_s^{k'}) < 0$ implies that

$$\Big(1 - \frac{p_{s^*}^{k'} p_s^k}{p_{s^*}^k p_s^{k'}}\Big)(x_s^k - x_s^{k'}) \geq 0,$$

for all $s \in S_1$. Hence $B \geq 0$.

□

4 Extension

4.1 Equal Consumptions

We have assumed that $x_s^k \neq x_{s'}^{k'}$ if $(k, s) \neq (k', s')$. We now relax this assumption. In this section, a *dataset* is a collection $(x^k, p^k)_{k=1}^K$ where for all k $x^k, p^k \in \mathbf{R}_{++}^S$.

When we allow for $x_s^k \neq x_{s'}^{k'}$, then there is a gap in our result: Axiom 2 is still sufficient for risk averse SEU rationality, but only necessary for SEU rationality with a differentiable utility function (the result in Varian (1983) on objective expected utility exhibits the same gap). A concave utility function is almost everywhere differentiable, so the gap is "small."

Definition 5. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* smooth SEU rational *if there is a vector* $\mu \in \mathbf{R}^S_{++}$ *with* $\sum_{s=1}^S \mu_s = 1$ *and a differentiable, concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k,

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x^k_s).$$

Theorem 3. *If a dataset satisfies Axiom 2 then it is SEU rational. If a dataset is smooth SEU rational, then it satisfies Axiom 2.*

5 Preliminaries

We shall use the following lemma, which is a version of the Theorem of the Alternative. This is Theorem 1.6.1 in Stoer and Witzgall (1970). We shall use it here in the cases where F is either the real or the rational numbers.

Lemma 1. *Let* A *be an* $m \times n$ *matrix,* B *be an* $l \times n$ *matrix, and* E *be an* $r \times n$ *matrix. Suppose that the entries of the matrices* A*,* B*, and* E *belong the a commutative ordered field* $\mathbf{F}$*. Exactly one of the following alternatives is true.*

1. *There is* $u \in \mathbf{F}^n$ *such that* $A \cdot u = 0$*,* $B \cdot u \geq 0$*,* $E \cdot u \gg 0$.
2. *There is* $\theta \in \mathbf{F}^r$*,* $\eta \in \mathbf{F}^l$*, and* $\pi \in \mathbf{F}^m$ *such that* $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$*;* $\pi > 0$ *and* $\eta \geq 0$.

We use the following notation in the proofs:

$$\mathcal{X} = \{x^k_s : k = 1, \ldots, K, s = 1, \ldots, S\}.$$

6 Proof of Theorem 1

We shall not prove the necessity direction. It has a simple proof, which follows along the lines of proving necessity in Theorem 2 in Section 7.

To prove sufficiency, we shall prove that there is a vector $\mu \in \mathbf{R}^S$ such that $\mu \gg 0$ and such that

$$x^{k'}_{s'} < x^k_s \Rightarrow \frac{\mu_{s'}}{p^{k'}_{s'}} \leq \frac{\mu_s}{p^k_s}. \tag{1}$$

We then define $f(x)$ by setting $f(x^k_s) = p^k_s / \mu_s$, and by interpolation everywhere else, so that f is a strictly decreasing function and positive everywhere (see the proof of Lemma 3 for an explicit argument). We then define $u(x) = \int_0^x f(t)dt$ to obtain a rationalization as desired: we have that

$$u'(x^k_s) = f(x^k_s) = \frac{p^k_s}{\mu_s},$$

so that the first-order condition for QL-SEU rationality is satisfied.

We use the following version of Farkas' Lemma, which directly follows from Lemma 1:

Lemma 2. *Let A be an $m \times n$ matrix. Exactly one of the following alternatives is true.*

1. *There is $\mu \in \mathbf{R}^n$ such that $A \cdot u \geq 0$ and $\mu \gg 0$.*
2. *There is $\theta \in \mathbf{R}^m$ such that $\theta \cdot A < 0$ and $\theta \geq 0$.*

Let A be a matrix with S columns, and $K^2S^2 - KS$ rows: one row for each set $\{s, s', k, k'\}$ with $(k, s) \neq (k', s')$, $s, s' = 1, \ldots, S$, and $k, k' = 1, \ldots, K$. The matrix A has all its entries zero, except as follows. For any states and observations s, s', k, k' with $(k, s) \neq (k', s')$, if $x^{k'}_{s'} < x^k_s$ then we have a row labeled (s, s', k, k') in which the entry for column s is $1/p^k_s$, and the entry for column s' is $-1/p^{k'}_{s'}$. The labeling (s, s', k, k') indicates in which row we have the positive entry $1/p^k_s$ and the negative entry $-1/p^{k'}_{s'}$. We denote the row by $r(s, s', k, k')$. Matrix A looks as follows:

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & \cdots & s & \cdots & s' & \cdots & S \end{array} \\
\begin{array}{c} \vdots \\ (s,s',k,k') \\ \vdots \end{array} &
\begin{bmatrix}
\vdots & & \vdots & & \vdots & & \vdots \\
0 & \cdots & \dfrac{1}{p^k_s} & \cdots & -\dfrac{1}{p^{k'}_{s'}} & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{bmatrix}
\end{array}
$$

By the definition of A, there is a solution to the conditions (1) if and only if there is $\mu \gg 0$ such that $A \cdot \mu \geq 0$. Suppose that there is no solution to the conditions (1). We shall prove that the data violate Axiom 1. By Lemma 2, there is a vector $\theta \geq 0$ with $\theta \cdot A < 0$. It could not be true that $\theta = 0$, so $\theta > 0$. Choose one such θ with the property that if θ' is another vector with the properties that $\theta' \geq 0$ and $\theta' \cdot A < 0$, then

$$\{r : \theta'_r > 0\} \not\subset \{r : \theta_r > 0\},$$

where $\subset$ means proper subset and θ_r denotes the entry of θ in row r. We can choose such a minimal θ because the number of rows in A is finite.

Claim 1. *There exists a sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^{n^*}_{i=1}$ that satisfies conditions (1) and (2) in Axiom 1.*

Proof. Fix a row $r(s, s', k, k')$ with $\theta_{r(s,s',k,k')} > 0$. Define $(s_1, s'_1, k_1, k'_1) = (s, s', k, k')$ and $\rho_1 = r(s_1, s'_1, k_1, k'_1)$. Observe that $x^{k'_1}_{s'_1} < x^{k_1}_{s_1}$.

The entry in row $r(s, s', k, k')$ and column s is $1/p^{k_1}_{s_1}$. Now, $\theta \cdot A < 0$ and $\theta_{\rho_1}/p^{k_1}_{s_1} > 0$, so there must exist a row $\rho_2 = r(s_2, s'_2, k_2, k'_2)$ with $s'_2 = s_1$ and $\theta_{\rho_3}/p^{k'_2}_{s'_2} < 0$. Note that $x^{k'_2}_{s'_2} < x^{k_2}_{s_2}$ and $s'_2 = s_1$.

In the column for s_2 of row ρ_2 we have $1/p_{s_2}^{k_2} > 0$, and $\theta_{\rho_2}/p_{s_2}^{k_2} > 0$, so again $\theta \cdot A < 0$ implies that there must exist a further row $\rho_3 = r(s_3, s_3', k_3, k_3')$ with $\theta_{\rho_3} > 0$ in which the entry for column $s_2 = s_3'$ is negative.

We can continue this process, whereby for each row $\rho_i = r(s_i, s_i', k_i, k_i')$ we identify another row $\rho_{i+1} = r(s_{i+1}, s_{i+1}', k_{i+1}, k_{i+1}')$ with $s_i = s_{i+1}'$. There is a finite number of states, so there must exist n^* such that $s_{n^*} = s_1'$. Note that the sequence of rows $(\rho_i)_{i=1}^{n^*}$ defines a sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ with $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ and $s_i = s_{i+1}'$, for $i = 1, \ldots, n^* - 1$ and $s_{n^*} = s_1'$. Thus each s either does not appear in the sequence, or it appears as both s_i and as s_{i+1}. Therefore each s equals s_i the for as many i as it equals s_i' in the sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$. □

Define

$$\begin{aligned}
\eta_{\rho_1} &= 1, \\
\eta_{\rho_2} &= \frac{p_{s_2'}^{k_2'}}{p_{s_1}^{k_1}}, \\
\eta_{\rho_3} &= \eta_{\rho_2} \frac{p_{s_3'}^{k_3'}}{p_{s_2}^{k_2}}, \\
&\vdots \\
\eta_{\rho_{i+1}} &= \eta_{\rho_i} \frac{p_{s_{i+1}'}^{k_{i+1}'}}{p_{s_i}^{k_i}}, \\
&\vdots \\
\eta_{\rho_{n^*}} &= \eta_{\rho_{n^*-1}} \frac{p_{s_{n^*}'}^{k_{n^*}'}}{p_{s_{n^*-1}}^{k_{n^*-1}}} = \frac{p_{s_2'}^{k_2'}}{p_{s_1}^{k_1}} \frac{p_{s_3'}^{k_3'}}{p_{s_2}^{k_2}} \cdots \frac{p_{s_{i+1}'}^{k_{i+1}'}}{p_{s_i}^{k_i}} \cdots \frac{p_{s_{n^*}'}^{k_{n^*}'}}{p_{s_{n^*-1}}^{k_{n^*-1}}}.
\end{aligned} \tag{2}$$

This defines η_{ρ_i} for all $i = 1, \ldots, n^*$. Define $\eta_r = 0$ for all other rows r. Define $y = \eta \cdot A$. Let R be the set of all rows in A and define $R^* = \{\rho_i : i = 1, \ldots, n^*\}$.

Claim 2. *(i) $y_s = 0$ if $s \notin \{s_1', s_{n^*}\}$ and (ii)* $y_{s_1'} = y_{s_{n^*}} = -\frac{1}{p_{s_1'}^{k_1'}} + \frac{1}{p_{s_{n^*}}^{k_{n^*}}} \prod_{i=1}^{n^*-1} \frac{p_{s_{i+1}'}^{k_{i+1}'}}{p_{s_i}^{k_i}}$.

Proof. Let $A(r, s)$ be the entry of A in row r and column s. First note that for each $\hat{s} \in S$,

$$\begin{aligned}
y_{\hat{s}} &= \sum_{\{r(s,s',k,k') \in R\}} \eta_{r(s,s',k,k')} A(r(s, s', k, k'), \hat{s}) \\
&= \sum_{\{r(s,s',k,k') \in R^* | s = \hat{s} \text{ or } s' = \hat{s}\}} \eta_{r(s,s',k,k')} A(r(s, s', k, k'), \hat{s}),
\end{aligned}$$

as $\eta_r = 0$ for all $r \notin R$.

First we show that $y_{\hat{s}} = 0$ if $\hat{s} \neq s_i$ and $\hat{s} \neq s_i'$ for all $i = 1, \ldots, n^*$. By the definition of A, $A(r(s, s', k, k'), \hat{s}) = 0$ if $s \neq \hat{s}$ and $s' \neq \hat{s}$. Therefore, if $\hat{s} \neq s_i$ and $\hat{s} \neq s_i'$ for all $i = 1, \ldots, n^*$, then $y_{\hat{s}} = 0$.

Note that $s_1 = s'_2$ and that

$$y_{s'_2} = y_{s_1} = \eta_{r_1}\frac{1}{p^{k_1}_{s_1}} - \eta_{r_2}\frac{1}{p^{k'_2}_{s'_2}} = \frac{1}{p^{k_1}_{s_1}} - \frac{p^{k'_2}_{s'_2}}{p^{k_1}_{s_1}}\frac{1}{p^{k'_2}_{s'_2}} = 0.$$

Similarly, $s_2 = s'_3$ and

$$y_{s'_3} = y_{s_2} = \frac{\eta_{r_2}}{p^{k_2}_{s_2}} - \frac{\eta_{r_3}}{p^{k'_3}_{s'_3}} = \eta_{r_2}\left(\frac{1}{p^{k_2}_{s_2}} - \frac{p^{k'_3}_{s'_3}}{p^{k_2}_{s_2}}\frac{1}{p^{k'_3}_{s'_3}}\right) = 0;$$

and so on: $s_i = s'_{i+1}$ and

$$y_{s'_{i+1}} = y_{s_i} = \frac{\eta_{r_i}}{p^{k_i}_{s_i}} - \frac{\eta_{r_{i+1}}}{p^{k'_{i+1}}_{s'_{i+1}}} = \eta_{r_i}\left(\frac{1}{p^{k_i}_{s_i}} - \frac{p^{k'_{i+1}}_{s'_{i+1}}}{p^{k_i}_{s_i}}\frac{1}{p^{k'_{i+1}}_{s'_{i+1}}}\right) = 0;$$

Continuing in this fashion, we obtain that $y_s = 0$ for all $s \neq s_1$.

Finally, we obtain the following:

$$y_{s_{n^*}} = y_{s'_1} = -\frac{\eta_{r_1}}{p^{k'_1}_{s'_1}} + \frac{\eta_{r_{n^*}}}{p^{k_{n^*}}_{s_{n^*}}} = -\frac{1}{p^{k'_1}_{s'_1}} + \frac{1}{p^{k_{n^*}}_{s_{n^*}}}\prod_{i=1}^{n^*-1}\frac{p^{k'_{i+1}}_{s'_{i+1}}}{p^{k_i}_{s_i}}$$

□

Claim 3. $y_{s'_1} < 0$

Proof. Suppose, towards a contradiction, that $y_{s'_1} \geq 0$. By Claim 2, $y_s = 0$ for all $s \neq s'_1$. Hence, $y \geq 0$. Note that by construction, if $\eta_r > 0$, then $\theta_r > 0$. Let

$$\delta = \min_{r\in R}\left\{\frac{\theta_r}{\eta_r} : \eta_r > 0\right\};$$

let $\theta' = \theta - \delta\eta$. Observe that $\theta \geq \theta' \geq 0$ and that $\theta' \cdot A = \theta \cdot A - \delta y < 0$, as $\theta \cdot A < 0$ and $y \geq 0$.

However, $\theta \geq \theta'$, and there is at least one row r for which $\theta'_r = 0$ and $\theta_r > 0$, so

$$\{r : \theta'_r > 0\} \subset \{r : \theta_r > 0\};$$

a contradiction of how we chose θ. □

Claim 4. *There exists a sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^{n^*}$ that satisfies conditions (1) and (2) in Axiom 1 but*

$$\prod_{i=1}^{n^*}\frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}} > 1.$$

Proof. By Claim 3, $y_{s'_1} < 0$. Then,

$$\frac{1}{p^{k_{n^*}}_{s_{n^*}}} \prod_{i=1}^{n^*-1} \frac{p^{k'_{i+1}}_{s'_{i+1}}}{p^{k_i}_{s_i}} < \frac{1}{p^{k'_1}_{s'_1}}$$

So,

$$1 < \frac{p^{k_{n^*}}_{s_{n^*}}}{p^{k'_1}_{s'_1}} \prod_{i=1}^{n^*-1} \frac{p^{k_i}_{s_i}}{p^{k'_{i+1}}_{s'_{i+1}}} = \prod_{i=1}^{n^*} \frac{p^{k_i}_{s_i}}{p^{k'_i}_{s'_i}}.$$

7 Proof of Theorem 2

The proof is based on using the first-order conditions for maximizing a utility with the SEU model over a budget set. Our first lemma ensures that we can without loss of generality restrict attention to first order conditions. The proof of the lemma is a matter of routine.

Lemma 3. *Let $(x^k, p^k)_{k=1}^K$ be a dataset. The following statements are equivalent:*

1. *$(x^k, p^k)_{k=1}^K$ is SEU rational.*
2. *$(x^k, p^k)_{k=1}^K$ is SEU rational with a continuously differentiable, strictly increasing and concave utility function.*
3. *There are strictly positive numbers v^k_s, λ^k, μ_s, for $s = 1, \ldots, S$ and $k = 1, \ldots, K$, such that*

$$\mu_s v^k_s = \lambda^k p^k_s$$
$$x^k_s > x^{k'}_{s'} \Rightarrow v^k_s < v^{k'}_{s'}.$$

Proof. That (2) implies (3) is immediate from the first-order conditions for maximizing a utility of the SEU model. We shall prove that (1) implies (2). Let $(x^k, p^k)_{k=1}^K$ be SEU rational. Let $\mu \in \mathbf{R}^S_{++}$ and $u : \mathbf{R}_+ \to \mathbf{R}$ be as in the definition of SEU rational data. Then (see, for example, Theorem 28.3 of Rockafellar (1997)), there are numbers $\lambda^k \geq 0$, $k = 1, \ldots, K$ such that

$$v^k_s = \frac{\lambda^k p^k_s}{\mu_s} \in \partial u(x^k_s),$$

for $s = 1, \ldots, S$ and $k = 1, \ldots, K$. In fact, it is easy to see that $\lambda^k > 0$, and therefore $v^k_s > 0$.

Enumerate elements in $\mathcal{X}$ in increasing order:

$$x^{k(1)}_{s(1)} < x^{k(2)}_{s(2)} < \ldots < x^{k(n)}_{s(n)}.$$

Note that it may be that $s(i) = s(j)$ or $k(i) = k(j)$ for some $i \neq j$.

Let $z_i = (x^{k(i)}_{s(i)} + x^{k(i+1)}_{s(i+1)})/2$, $i = 1, \dots, n-1$; $z_0 = 0$, and $z_n = x^{k(n)}_{s(n)} + 1$. Let $f : \mathbf{R}_{++} \to \mathbf{R}_{++}$ be defined as

$$f(z) = \begin{cases} v^{k(i)}_{s(i)} & \text{if } z \in (z_{i-1}, z_i], \\ v^{k(i)}_{s(i)}(\frac{z_n}{z^2})^2 & \text{if } z > z_n. \end{cases}$$

Since u is concave, $v^{k(i)}_{s(i)} \geq v^{k(i+1)}_{s(i+1)}$. Therefore $f > 0$ and f is strictly decreasing. Let $\varepsilon > 0$ be such that

$$\varepsilon \leq \min\{z_j - x^{k(i)}_{s(i)} : i, j = 1 \dots, n\}.$$

Note that f is constant and equal to $v^{k_i}_{s_i}$ on any interval $(x^{k_i}_{s_i} - \varepsilon, x^{k_i}_{s_i} + \varepsilon)$.

Let $\psi : \mathbf{R} \to \mathbf{R}$ be an infinitely differentiable function such that (a) $\psi(x) \geq 0$ for every $x \in \mathbf{R}$; (b) $\psi(x) = 0$ when $|x| \geq \varepsilon$, and (c) $\int_{\mathbf{R}} \psi = 1$. (A *mollifier* .) For example, we can choose

$$\psi(x) = \begin{cases} \frac{1}{C} e^{-1/(1-(x/\varepsilon)^2)}, & \text{if } |x| < \varepsilon \\ 0 & \text{otherwise,} \end{cases}$$

for a suitable normalizing factor C.

Finally, define the function $u^* : \mathbf{R}_+ \to \mathbf{R}$ by

$$u^*(x) = \int_{\mathbf{R}} f(x-y)\psi(y)dy.$$

Then it follows from standard arguments that u^* is continuously differentiable, strictly increasing, and concave.

Since f is constant and equal to $v^{k_i}_{s_i}$ on $(x^{k_i}_{s_i} - \varepsilon, x^{k_i}_{s_i} + \varepsilon)$, the derivative at x^k_s is

$$Du^*(x^k_s) = \int_{-\varepsilon}^{\varepsilon} f'(x-y)\psi(y)dy = \int_{-\varepsilon}^{\varepsilon} v^k_s \psi(y)dy = v^k_s,$$

so that x^k_s satisfies the first order condition for maximizing

$$\sum_{s=1}^{S} \mu_s u^*(x_s)$$

over the budget set $\{y \in \mathbf{R}^S_+ : p^k \cdot y \leq p^k \cdot x^k\}$. Hence μ and u^* SEU rationalize the data.

Finally, we prove that (3) implies (2). The proof is analogous to the proof that (1) implies (2). Given numbers v^k_s, λ^k and μ_s as in (3), let $\mu'_s = \mu_s / \sum_{\hat{s}} \mu_{\hat{s}}$ and $\theta^k = \lambda^k / \sum_{\hat{s}} \mu_{\hat{s}}$. We obtain that $\mu'_s v^k_s = \theta^k p^k_s$. Define f from v^k_s as above. Then $f > 0$ and f is strictly decreasing. Defining $u^*(x) = \int_{-\infty}^{x} f(t)dt$ as above ensures that μ' and u^* SEU rationalize the data.

Obviously (2) implies (1). $\square$

7.1 Necessity

Lemma 4. *If a dataset* $(x^k, p^k)_{k=1}^k$ *is SEU rational, then it satisfies Axiom 2*

Proof. By Lemma 3, if a dataset is SEU rational then there is a continuously differentiable and concave rationalization u and a strictly positive solution v_s^k, λ^k, μ_s to the system in Statement (3) of Lemma 3 with $u'(x_s^k) = v_s^k$. Let $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ be a sequence satisfying the three conditions in Axiom 2. Then $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$, so

$$1 \geq \frac{u'(x_{s_i}^{k_i})}{u'(x_{s_i'}^{k_i'})} = \frac{\lambda^{k_i}\mu_{s_i'}p_{s_i}^{k_i}}{\lambda^{k_i'}\mu_{s_i}p_{s_i'}^{k_i'}}.$$

Thus,

$$1 \geq \prod_{i=1}^n \frac{u'(x_{s_i}^{k_i})}{u'(x_{s_i'}^{k_i'})} = \prod_{i=1}^n \frac{\lambda^{k_i}\mu_{s_i'}p_{s_i}^{k_i}}{\lambda^{k_i'}\mu_{s_i}p_{s_i'}^{k_i'}} = \prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}},$$

as the sequence satisfies (2) and (3) of Axiom 2; and hence the numbers λ^k and μ_s appear the same number of times in the denominator as in the numerator of this product. □

7.2 Sufficiency

We proceed to prove the sufficiency direction. Sufficiency follows from the following lemmas. We know from Lemma 3 that it suffices to find a solution to the first order conditions. Lemma 5 establishes that Axiom 2 is sufficient when the logarithms of the prices are rational numbers. The role of rational logarithms comes from our use of a version of Farkas's Lemma. Lemma 6 says that we can approximate any data satisfying Axiom 2 with a dataset for which the logs of prices are rational and for which Axiom 2 is satisfied. Finally, Lemma 7 establishes the result. It is worth mentioning that we cannot use Lemma 6 and an approximate solution to obtain a limiting solution.

Lemma 5. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy Axiom 2. Suppose that* $\log(p_s^k) \in \mathbf{Q}$ *for all* k *and* s*. Then there are numbers* v_s^k, λ^k, μ_s*, for* $s = 1, \dots, S$ *and* $k = 1, \dots, K$ *satisfying* (3) *in Lemma 3.*

Lemma 6. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy Axiom 2. Then for all positive numbers* $\bar{\varepsilon}$*, there exists* $q_s^k \in [p_s^k - \bar{\varepsilon}, p_s^k]$ *for all* $s \in S$ *and* $k \in K$ *such that* $\log q_s^k \in \mathbf{Q}$ *and the data* $(x^k, q^k)_{k=1}^k$ *satisfy Axiom 2.*

Lemma 7. *Let data* $(x^k, p^k)_{k=1}^k$ *satisfy Axiom 2. Then there are numbers* v_s^k, λ^k, μ_s*, for* $s = 1, \dots, S$ *and* $k = 1, \dots, K$ *satisfying* (3) *in Lemma 3.*

To prove Lemmas 5 and 7, we use Lemma 1 and the following lemma.

Lemma 8. *Let* A *be an* $m \times n$ *matrix,* B *be an* $l \times n$ *matrix, and* E *be an* $r \times n$ *matrix. Suppose that the entries of the matrices* A*,* B*, and* E *are rational numbers. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{R}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, and $E \cdot u \gg 0$.*

2. *There is $\theta \in \mathbf{Q}^r$, $\eta \in \mathbf{Q}^l$, and $\pi \in \mathbf{Q}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\theta > 0$ and $\eta \geq 0$.*

Lemma 8 follows from Lemma 1: see Border (2013) or Chambers and Echenique (2011).

7.2.1 Proof of Lemma 5

We linearize the equation in System (3) of Lemma 3. The result is:

$$\log v_s^k + \log \mu_s - \log \lambda^k - \log p_s^k = 0, \tag{3}$$

$$x_s^k > x_{s'}^{k'} \Rightarrow \log v_s^k \leq \log v_{s'}^{k'} \tag{4}$$

In the system comprised by (3) and (4), the unknowns are the real numbers $\log v_s^k$, $\log \mu_s$, $\log \lambda^k$, $k = 1, \ldots, K$ and $s = 1, \ldots, S$.

First, we are going to write the system of inequalities (3) and (4) in matrix form.

A system of linear inequalities

We shall define a matrix A such that there are positive numbers v_s^k, λ^k, μ_s the logs of which satisfy Equation (3) if and only if there is a solution $u \in \mathbf{R}^{K \times S + K + S + 1}$ to the system of equations

$$A \cdot u = 0,$$

and for which the last component of u is strictly positive.

Let A be a matrix with $K \times S$ rows and $K \times S + S + K + 1$ columns, defined as follows: We have one row for every pair (k, s); one column for every pair (k, s); one column for each k; one column for every s; and one last column. In the row corresponding to (k, s) the matrix has zeroes everywhere with the following exceptions: it has a 1 in the column for (k, s); it has a 1 in the column for s; it has a -1 in the column for k; and $-\log p_s^k$ in the very last column.

Matrix A looks as follows:

$$\begin{array}{c} \\ {\scriptstyle (1,1)} \\ \vdots \\ {\scriptstyle (k,s)} \\ \vdots \\ {\scriptstyle (K,S)} \end{array}
\begin{array}{c}
\begin{array}{ccccc|ccccc|ccccc|c}
{\scriptstyle (1,1)} & {\scriptstyle \cdots} & {\scriptstyle (k,s)} & {\scriptstyle \cdots} & {\scriptstyle (K,S)} & {\scriptstyle 1} & {\scriptstyle \cdots} & {\scriptstyle s} & {\scriptstyle \cdots} & {\scriptstyle S} & {\scriptstyle 1} & {\scriptstyle \cdots} & {\scriptstyle k} & {\scriptstyle \cdots} & {\scriptstyle K} & {\scriptstyle p}
\end{array} \\
\left[\begin{array}{ccccc|ccccc|ccccc|c}
1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & -1 & \cdots & 0 & \cdots & 0 & -\log p_1^1 \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & -1 & \cdots & 0 & -\log p_s^k \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & -1 & -\log p_S^K
\end{array}\right]
\end{array}$$

Consider the system $A \cdot u = 0$. If there are numbers solving Equation (3), then these define a solution $u \in \mathbf{R}^{K \times S + S + K + 1}$ for which the last component is 1. If, on the other hand, there is a solution $u \in \mathbf{R}^{K\times S+S+K+1}$ to the system $A \cdot u = 0$ in which the last component is strictly positive, then by dividing through by the last component of u we obtain numbers that solve Equation (3).

In second place, we write the system of inequalities (4) in matrix form. Let B be a matrix B with $|\mathcal{X}|(|\mathcal{X}|-1)/2$ rows and $K \times S + S + K + 1$ columns. Define B as follows: One row for every pair $x, x' \in \mathcal{X}$ with $x > x'$; in the row corresponding to $x, x' \in \mathcal{X}$ with $x > x'$ we have zeroes everywhere with the exception of a -1 in the column for (k, s) such that $x = x^k_s$ and a 1 in the column for (k', s') such that $x' = x^{k'}_{s'}$. These define $|\mathcal{X}|(|\mathcal{X}|-1)/2$ rows.

In third place, we have a matrix E that captures the requirement that the last component of a solution be strictly positive. The matrix E has a single row and $K\times S+S+K+1$ columns. It has zeroes everywhere except for 1 in the last column.

To sum up, there is a solution to system (3) and (4) if and only if there is a vector $u \in \mathbf{R}^{K\times S+S+K+1}$ that solves the system of equations and linear inequalities

$$S1: \begin{cases} A \cdot u = 0, \\ B \cdot u \geq 0, \\ E \cdot u \gg 0. \end{cases}$$

Note that $E \cdot u$ is a scalar, so the last inequality is the same as $E \cdot u > 0$.

<u>Theorem of the Alternative</u>

The entries of A, B, and E are either 0, 1 or -1, with the exception of the last column of A. Under the hypothesis of the lemma we are proving, the last column consists of rational numbers. By Lemma 8, then, there is such a solution u to $S1$ if and only if there is no vector $(\theta, \eta, \pi) \in \mathbf{Q}^{K\times S+(|\mathcal{X}|(|\mathcal{X}|-1)/2)+1}$ that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

In the following, we shall prove that the non-existence of a solution u implies that the data must violate Axiom 2. Suppose then that there is no solution u and let (θ, η, π) be a rational vector as above, solving system $S2$.

By multiplying (θ, η, π) by any positive integer we obtain new vectors that solve $S2$, so we can take (θ, η, π) to be integer vectors.

Henceforth, we use the following notational convention: For a matrix D with $K \times S + S + K + 1$ columns, write D_1 for the submatrix of D corresponding to the first $K \times S$

columns; let D_2 be the submatrix corresponding to the following S columns; D_3 correspond to the next K columns; and D_4 to the last column. Thus, $D = [D_1|D_2|D_3|D_4]$.

Claim 5. *(i) $\theta \cdot A_1 + \eta \cdot B_1 = 0$; (ii) $\theta \cdot A_2 = 0$; (iii) $\theta \cdot A_3 = 0$; and (iv) $\theta \cdot A_4 + \pi \cdot E_4 = 0$.*

Proof. Since $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$, then $\theta \cdot A_i + \eta \cdot B_i + \pi \cdot E_i = 0$ for all $i = 1, \ldots, 4$. Moreover, since B_2, B_3, B_4, E_1, E_2, and E_3 are zero matrices, we obtain the claim. $\square$

For convenience, we transform the matrices A and B using θ and η.

<u>Transform the matrices A and B</u>

Lets define a matrix A^* from A by letting A^* have the same number of columns as A and including

1. θ_r copies of the rth row when $\theta_r > 0$;
2. omitting row r when $\theta_r = 0$;
3. and θ_r copies of the rth row multiplied by -1 when $\theta_r < 0$.

We refer to rows that are copies of some r with $\theta_r > 0$ as *original* rows, and to those that are copies of some r with $\theta_r < 0$ as *converted* rows.

Similarly, we define the matrix B^* from B by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

Claim 6. *For any (k, s), all the entries in the column for (k, s) in A_1^* are of the same sign.*

Proof. By definition of A, the column for (k, s) will have zero in all its entries with the exception of the row for (k, s). In A^*, for each (k, s), there are three mutually exclusive possibilities: the row for (k, s) in A can (i) not appear in A^*, (ii) it can appear as original, or (iii) it can appear as converted. This shows the claim. $\square$

Claim 7. *There exists a sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ that satisfies (1) in Axiom 2.*

Proof. We define such a sequence by induction. Let $B^1 = B^*$. Given B^i, define B^{i+1} as follows.

Denote by $>^i$ the binary relation on $\mathcal{X}$ defined by $z >^i z'$ if $z > z'$ and there is at least one copy of the row corresponding to $z > z'$ in B^i. The binary relation $>^i$ cannot exhibit cycles because $>^i \subseteq >$. There is therefore at least one sequence $z_1^i, \ldots z_{L_i}^i$ in $\mathcal{X}$ such that $z_j^i >^i z_{j+1}^i$ for all $j = 1, \ldots, L_i - 1$ and with the property that there is no $z \in \mathcal{X}$ with $z >^i z_1^i$ or $z_{L_i}^i >^i z$.

Let the matrix B^{i+1} be defined as the matrix obtained from B^i by omitting one copy of the row corresponding to $z_j^i > z_{j+1}^i$, for all $j = 1, \ldots L_i - 1$.

The matrix B^{i+1} has strictly fewer rows than B^i. There is therefore n^* for which B^{n^*+1} would have no rows. The matrix B^{n^*} has rows, and the procedure of omitting rows from B^{n^*} will remove all rows of B^{n^*}.

Define a sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^{n^*}$ by letting $x^{k_i}_{s_i} = z^i_1$ and $x^{k'_i}_{s'_i} = z^i_{L_i}$. Note that, as a result, $x^{k_i}_{s_i} > x^{k'_i}_{s'_i}$ for all i. Therefore the sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^{n^*}$ satisfies condition (1) in Axiom 2. □

We shall use the sequence of pairs $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})_{i=1}^{n^*}$ as our candidate violation of Axiom 2.

Consider a sequence of matrices A^i, $i = 1, \ldots, n^*$ defined as follows. Let $A^1 = A^*$, and

$$C^1 = \begin{bmatrix} A^1 \\ B^1 \end{bmatrix}.$$

Observe that the rows of C^1 add to the null vector by Claim 5.

We shall proceed by induction. Suppose that A^i has been defined, and that the rows of

$$C^i = \begin{bmatrix} A^i \\ B^i \end{bmatrix}$$

add to the null vector.

Recall the definition of the sequence

$$x^{k_i}_{s_i} = z^i_1 > \ldots > z^i_{L_i} = x^{k'_i}_{s'_i}.$$

There is no $z \in \mathcal{X}$ with $z >^i z^i_1$ or $z^i_{L_i} >^i z$, so in order for the rows of C^i to add to zero there must be a -1 in A^i_1 in the column corresponding to (k'_i, s'_i) and a 1 in A^i_1 in the column corresponding to (k_i, s_i). Let r_i be a row in A^i corresponding to (k_i, s_i), and r'_i be a row corresponding to (k'_i, s'_i). The existence of a -1 in A^i_1 in the column corresponding to (k'_i, s'_i), and a 1 in A^i_1 in the column corresponding to (k_i, s_i), ensures that r_i and r'_i exist. Note that the row r'_i is a converted row while r_i is original. Let A^{i+1} be defined from A^i by deleting the two rows, r_i and r'_i.

Claim 8. *The sum of r_i, r'_i, and the rows of B^i which are deleted when forming B^{i+1} (corresponding to the pairs $z^i_j > z^i_{j+1}$, $j = 1, \ldots, L_i - 1$) add to the null vector.*

Proof. Recall that $z^i_j >^i z^i_{j+1}$ for all $j = 1, \ldots, L_i - 1$. So when we add the rows corresponding to $z^i_j >^i z^i_{j+1}$ and $z^i_{j+1} >^i z^i_{j+2}$, then the entries in the column for (k, s) with $x^k_s = z^i_{j+1}$ cancel out and the sum is zero in that entry. Thus, when we add the rows of B^i that are not in B^{i+1} we obtain a vector that is 0 everywhere except the columns corresponding to z^i_1 and $z^i_{L_i}$. This vector cancels out with $r_i + r'_i$, by definition of r_i and $'_i$. □

Since the rows of C^i add up to the null vector, and A^{i+1} and B^{i+1} are obtained from A^i and B^i by removing a collection of rows that add up to zero, then the rows of C^{i+1} must add up to zero as well.

Claim 9. *The matrix A^* can be partitioned into pairs of rows as follows:*

$$A^* = \begin{bmatrix} r_1 \\ r_1' \\ \vdots \\ r_i \\ r_i' \\ \vdots \\ r_{n^*} \\ r_{n^*}', \end{bmatrix}$$

in which the rows r_i' are converted and the rows r_i are original.

Proof. For each i, A^{i+1} differs from A^i in that the rows r_i and r_i' are removed from A^i to form A^{i+1}. We shall prove that A^* is composed of the $2n^*$ rows r_i, r_i'.

By way of contradiction, suppose that there exist rows left after removing r_{n^*} and r_{n^*}'. Then, by the argument above, the rows of the matrix C^{n^*+1} must add to the null vector. If there are rows left, then the matrix C^{n^*+1} is well defined.

By definition of the sequence B^i, however, B^{n^*+1} is an empty matrix. Hence, rows remaining in $A_1^{n^*+1}$ must add up to zero. By Claim 6, the entries of a column (k, s) of A^* are always of the same sign. Moreover, each row of A^* has a non-zero element in the first $K \times S$ columns. Therefore, no subset of the columns of A_1^* can sum to the null vector. $\square$

Claim 10. *(i) For any k and s, if $x_{s_i}^{k_i} = x_s^k$ for some i, then the row r_i corresponding to (k, s) appears as original in A^*. Similarly, if $x_{s_i'}^{k_i'} = x_s^k$ for some i, then the row corresponding to (k, s) appears converted in A^*.*
(ii) If the row corresponding to (k, s) appears as original in A^*, then there is some i with $x_{s_i}^{k_i} = x_s^k$. Similarly, if the row corresponding to (k, s) appears converted in A^*, then there is i with $x_{s_i'}^{k_i'} = x_s^k$.

Proof. (i) is true by definition of $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})$. (ii) is immediate from Claim 9 because if the row corresponding to (k, s) appears original in A^* then it equals r_i for some i, and then $x_s^k = x_{s_i}^{k_i}$. Similarly when the row appears converted. $\square$

Claim 11. *The sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ satisfies (2) and (3) in Axiom 2.*

Proof. By Claim 5 (ii), the rows of A_2^* add up to zero. Therefore, the number of times that s appears in an original row equals the number of times that it appears in a converted row. By Claim 10, then, the number of times s appears as s_i equals the number of times it appears as s'_i. Therefore condition (2) in the axiom is satisfied.

Similarly, by Claim 5 (iii), the rows of A_3^* add to the null vector. Therefore, the number of times that a state k appears in an original row equals the number of times that it appears in a converted row. By Claim 10, then, the number of times that k appears as k_i equals the number of times it appears as k'_i. Therefore condition (3) in the axiom is satisfied. □

Finally, in the following, we show that

$$\prod_{i=1}^{n^*} \frac{p_{s_i}^{k_i}}{p_{s'_i}^{k'_i}} > 1,$$

which finishes the proof of Lemma 5 as the sequence $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^{n^*}$ would then exhibit a violation of Axiom 2.

Claim 12. $\prod_{i=1}^{n^*} \frac{p_{s_i}^{k_i}}{p_{s'_i}^{k'_i}} > 1$.

Proof. By Claim 5 (iv) and the fact that the submatrix E_4 equals the scalar 1, we obtain

$$0 = \theta \cdot A_4 + \pi E_4 = (\sum_{i=1}^{n^*}(r_i + r'_i))_4 + \pi,$$

where $(\sum_{i=1}^{n^*}(r_i + r'_i))_4$ is the (scalar) sum of the entries of A_4^*. Recall that $-\log p_{s_i}^{k_i}$ is the last entry of row r_i and that $\log p_{s'_i}^{k'_i}$ is the last entry of row r'_i, as r'_i is converted and r_i original. Therefore the sum of the rows of A_4^* are $\sum_{i=1}^{n^*} \log(p_{s'_i}^{k'_i}/p_{s_i}^{k_i})$. Then,

$$\sum_{i=1}^{n^*} \log(p_{s'_i}^{k'_i}/p_{s_i}^{k_i}) = -\pi < 0.$$

Thus

$$\prod_{i=1}^{n^*} \frac{p_{s_i}^{k_i}}{p_{s'_i}^{k'_i}} > 1.$$

□

7.2.2 Proof of Lemma 6

For each sequence $\sigma = (x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^{n}$ that satisfies conditions (1), (2), and (3) in Axiom 2, and each pair $x_s^k > x_{s'}^{k'}$, define $t_\sigma(x_s^k, x_{s'}^{k'})$ to be the number of times the pair $(x_s^k, x_{s'}^{k'})$

appears in the sequence σ. Note that t_σ is a $\frac{KS(K-1)(S-1)}{2}$-dimensional non-negative integer vector. Define

$$T \;=\; \left\{ t_\sigma \in \mathbf{N}^{\frac{KS(K-1)(S-1)}{2}} | \sigma \text{ satisfies (1), (2), (3) in Axiom 2 } \right\}.$$

The set T depends only on $(x^k)_{k=1}^K$ in the data set $(x^k, p^k)_{k=1}^K$.

For each pair $x_s^k > x_{s'}^{k'}$, define

$$\hat{\delta}(x_s^k, x_{s'}^{k'}) = \log \frac{p_s^k}{p_{s'}^{k'}}.$$

Then, $\hat{\delta}$ is a $\frac{KS(K-1)(S-1)}{2}$-dimensional real-valued vector.

If $\sigma = (x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$, then

$$\hat{\delta} \cdot t_\sigma = \sum_{(x_s^k, x_{s'}^{k'}) \in \sigma} \hat{\delta}(x_s^k, x_{s'}^{k'}) t_\sigma(x_s^k, x_{s'}^{k'}) = \log \Big(\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \Big).$$

So the data satisfy Axiom 2 if and only if $t \cdot \hat{\delta} \leq 0$ for all $t \in T$.

Enumerate elements in $\mathcal{X}$ in increasing order:

$$x_{s(1)}^{k(1)} < x_{s(2)}^{k(2)} < \cdots < x_{s(N)}^{k(N)}.$$

Fix arbitrary numbers $\underline{\xi}, \bar{\xi} \in (0,1)$ with $\underline{\xi} < \bar{\xi}$. Due to the denseness of the rational numbers, and the continuity of the exponential function, there exists a positive number $\varepsilon(1)$ such that $\log(p_{s(1)}^{k(1)}\varepsilon(1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon(1) < 1$; Given $\varepsilon(1)$, there exists a positive $\varepsilon(2)$ such that $\log(p_{s(2)}^{k(2)}\varepsilon(2)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon(2)$ and $\varepsilon(2)/\varepsilon(1) < \bar{\xi}$. More generally, when $\varepsilon(n)$ has been defined, let $\varepsilon(n+1) > 0$ be such that $\log(p_{s(n+1)}^{k(n+1)}\varepsilon(n+1)) \in \mathbf{Q}$, $\underline{\xi} < \varepsilon(n+1)$ and $\varepsilon(n+1)/\varepsilon(n) < \bar{\xi}$.

In this way have defined $(\varepsilon(n))_{n=1}^N$. Let $q_s^k = p_s^k \varepsilon(n)$. The claim is that the data $(x^k, q^k)_{k=1}^K$ satisfy Axiom 2. Let δ^* be defined from $(q^k)_{k=1}^K$ in the same manner as $\hat{\delta}$ was defined from $(p^k)_{k=1}^K$.

For each pair $x_s^k > x_{s'}^{k'}$, if n and m are such that $x_s^k = x_{s(n)}^{k(n)}$ and $x_{s'}^{k'} = x_{s(m)}^{k(m)}$, then $n > m$. By definition of ε, $\varepsilon(n)/\varepsilon(m) < \bar{\xi} < 1$. Hence,

$$\delta^*(x_s^k, x_{s'}^{k'}) = \log \frac{p_s^k \varepsilon(n)}{p_{s'}^{k'}\varepsilon(m)} < \log \frac{p_s^k}{p_{s'}^{k'}} + \log \bar{\xi} < \log \frac{p_s^k}{p_{s'}^{k'}} = \hat{\delta}(x_s^k, x_{s'}^{k'}).$$

Thus, for all $t \in T$,

$$\delta^* \cdot t \leq \hat{\delta} \cdot t \leq 0,$$

as $t \geq 0$ and the data $(x^k, p^k)_{k=1}^K$ satisfy Axiom 2. Thus the data $(x^k, q^k)_{k=1}^K$ satisfy Axiom 2.

Note that $\underline{\xi} < \varepsilon(n)$ for all n. So that by choosing $\underline{\xi}$ close enough to 1 we can take the prices (q^k) to be as close to (p^k) as desired.

7.2.3 Proof of Lemma 7

Consider the system comprised by (3) and (4) in the proof of Lemma 5. Let A, B, and E be constructed from the data as in the proof of Lemma 5. The difference with respect to Lemma 5 is that now the entries of A_4 may not be rational. Note that the entries of E, B, and A_i, $i = 1, 2, 3$ are rational.

Suppose, towards a contradiction, that there is no solution to the system comprised by (3) and (4). Then, by the argument in the proof of Lemma 5 there is no solution to System $S1$. By Lemma 1 with $\mathbf{F} = \mathbf{R}$, there is a real vector (θ, η, π) such that

$$\theta \cdot A + \eta \cdot B + \pi \cdot E = 0 \text{ and } \eta \geq 0, \pi > 0.$$

Recall that $B_4 = 0$ and $E_4 = 1$, so we obtain that $\theta \cdot A_4 + \pi = 0$.

Let $(q^k)_{k=1}^K$ be vectors of prices such that the dataset $(x^k, q^k)_{k=1}^K$ satisfies Axiom 2 and $\log q_s^k \in \mathbf{Q}$ for all k and s. (Such $(q^k)_{k=1}^K$ exists by Lemma 6.) Construct matrices A', B', and E' from this dataset in the same way as A, B, and E is constructed in the proof of Lemma 5. Note that only the prices are different in (x^k, q^k) compared to (x^k, p^k). So $E' = E$, $B' = B$ and $A'_i = A_i$ for $i = 1, 2, 3$. Since only prices q^k are different in this dataset, only A'_4 may be different from A_4.

By Lemma 6, we can choose prices q^k such that $|\theta \cdot A'_4 - \theta \cdot A_4| < \pi/2$. We have shown that $\theta \cdot A_4 = -\pi$, so the choice of prices q^k guarantees that $\theta \cdot A'_4 < 0$. Let $\pi' = -\theta \cdot A'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \pi' E_i = 0$ for $i = 1, 2, 3$, as (θ, η, π) solves system $S2$ for matrices A, B and E, and $A'_i = A_i$, $B'_i = B_i$ and $E_i = 0$ for $i = 1, 2, 3$. Finally, $B_4 = 0$ so

$$\theta \cdot A'_4 + \eta \cdot B'_4 + \pi' E_4 = \theta \cdot A'_4 + \pi' = 0.$$

We also have that $\eta \geq 0$ and $\pi' > 0$. Therefore θ, η, and π' constitute a solution $S2$ for matrices A', B', and E'.

By Lemma 1 we know then that there is no solution to $S1$ for matrices A', B', and E', so there is no solution to the system comprised by (3) and (4) in the proof of Lemma 5. However, this contradicts Lemma 5 because the data (x^k, q^k) satisfies Axiom 2 and $\log q_s^k \in \mathbf{Q}$ for all $k = 1, \ldots K$ and $s = 1, \ldots, S$.

8 Proof of Theorem 3

The second statement in the theorem follows from Lemma 3 and the proof of Lemma 4. We proceed to prove the first statement in the theorem. Assume then that $(x^k, p^k)_{k=1}^K$ is a dataset that satisfies Axiom 2.

Recall that $\mathcal{X} = \{x_s^k : k = 1, \ldots, K, s = 1, \ldots, S\}$. Let $\varepsilon > 0$ be s.t.

$$\varepsilon < \min\{|x - x'| : x, x' \in \mathcal{X}, x \neq x'\}.$$

Let $\alpha(x) = \{(k, s) : x = x_s^k\}$ for $x \in \mathcal{X}$.

We shall define a new dataset for which consumptions are not equal, but that still satisfies Axiom 2. Let $(\hat{x}^k, p^k)_{k=1}^K$ be a dataset with the same prices as in $(x^k, p^k)_{k=1}^K$; in which $(\hat{x}^k)_{k=1}^K$ is chosen such that (a) $\hat{x}_s^k \neq \hat{x}_{s'}^{k'}$ when $(k, s) \neq (k', s')$; and (b) for all $x \in \mathcal{X}$

$$|\hat{x}_s^k - x| < \varepsilon,$$

for all $(k, s) \in \alpha(x)$.

Observe that, with this definition of data $(\hat{x}^k, p^k)_{k=1}^K$, if $\hat{x}_s^k > \hat{x}_{s'}^{k'}$ then $x_s^k \geq x_{s'}^{k'}$. The reason is that, either there is x for which $(k, s) \in \alpha(x)$ and $(k', s') \in \alpha(x)$, in which case $x_s^k \geq x_{s'}^{k'}$ because $x = x_s^k = x_{s'}^{k'}$; or there is no x and x', with $x \neq x'$, in which $(k, s) \in \alpha(x)$ and $(k', s') \in \alpha(x')$, which implies that $x > x'$ and thus that $x_s^k > x_{s'}^{k'}$.

With this definition of data, if $(\hat{x}_{s_i}^{k_i}, \hat{x}_{s_i'}^{k_i'})_{i=1}^n$ is a sequence of pairs from dataset $(\hat{x}^k, p^k)_{k=1}^K$ satisfying (1), (2), and (3) in Axiom 2, then $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ is a sequence of pairs from dataset $(x^k, p^k)_{k=1}^K$ that also satisfies (1), (2), and (3) in Axiom 2. By hypothesis, $(x^k, p^k)_{k=1}^K$ satisfy Axiom 2, so $(\hat{x}^k, p^k)_{k=1}^K$ satisfy Axiom 2.

Since $(\hat{x}^k, p^k)_{k=1}^K$ satisfies that $x_s^k \neq x_{s'}^{k'}$ if $(k, s) \neq (k', s')$, and Axiom 2, then Lemma 7 implies that there are strictly positive numbers $\hat{v}_s^k$, λ^k, μ_s, for $s = 1, \ldots, S$ and $k = 1, \ldots, K$, such that

$$\begin{gathered} \mu_s \hat{v}_s^k = \lambda^k p_s^k \\ \hat{x}_s^k > \hat{x}_{s'}^{k'} \Rightarrow \hat{v}_s^k < \hat{v}_{s'}^{k'}. \end{gathered}$$

Define the correspondence $v' : \mathcal{X} \to \mathbf{R}_+$ by

$$v'(x) = \left[\inf\{\hat{v}_s^k(k, s) \in \alpha(x)\}, \sup\{\hat{v}_s^k(k, s) \in \alpha(x)\}\right].$$

Note that if $x > x'$ then $\hat{v}_s^k < \hat{v}_{s'}^{k'}$ for all $(k, s) \in \alpha(x)$ and all $(k's') \in \alpha(x')$. So as a result of the definition of v', if $x > x'$ then $\sup v'(x) < \inf v'(x')$.

Let $v : \mathbf{R}_+ \to \mathbf{R}_+$ be

$$v(x) = \{\inf v'(\tilde{x}) : \tilde{x} \in \mathcal{X}, \tilde{x} \leq x\}$$

for $x \geq \inf \mathcal{X}$; and $v(x) = \{\sup v'(\tilde{x}) : \tilde{x} \in \mathcal{X}\}$ for $x < \inf \mathcal{X}$. The correspondence v is monotone. There is therefore a concave function $u : \mathbf{R}_+ \to \mathbf{R}$ such that

$$\partial u(x) = v(x)$$

for all x (See Rockafellar (1997)).

In particular, for all $x \in \mathcal{X}$ and all $(k,s) \in \alpha(x)$ we have $\hat{v}_s^k \in \partial u(x)$. Since $\mu_s \hat{v}_s^k = \lambda^k p_s^k$, we have

$$\frac{\lambda^k p_s^k}{\mu_s} \in \partial u(x_s^k).$$

Hence the first-order conditions for SEU maximization are satisfied at x_s^k.

References

Border, K. C. (2013): "Alternative linear inequalities," Mimeo, Caltech.

Chambers, C. P. and F. Echenique (2011): "On the consistency of data with bargaining theories," Forthcoming, Theoretical Economics.

Echenique, F. and K. Saito (2013): "Revealed Preference Foundation for Exponential Discounting Utility Model," Mimeo, Caltech.

Ellsberg, D. (1961): "Risk, Ambiguity, and the Savage Axioms," *The Quarterly Journal of Economics*, 75, 643–669.

Epstein, L. G. (2000): "Are probabilities used in markets?" *Journal of Economic Theory*, 91, 86–90.

Green, R. C. and S. Srivastava (1986): "Expected Utility Maximization and Demand Behavior," *Journal of Economic Theory*, 38, 313–323.

Kreps, D. M. (1988): *Notes on the Theory of Choice*, Colorado: Westview Press.

Rockafellar, R. T. (1997): *Convex analysis*, Princeton university press.

Savage, L. J. (1954): *The Foundations of Statistics*, New York: Wiley.

Stoer, J. and C. Witzgall (1970): *Convexity and optimization in finite dimensions*, Springer-Verlag Berlin.

Varian, H. R. (1983): "Nonparametric Tests of Models of Investor Behavior," *Journal of Financial and Quantitative Analysis*, 18, 269–278.

——— (1988): "Estimating Risk Aversion from Arrow-Debreu Portfolio Choice," *Econometrica*, 56, 973–979.